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ON BASICITY OF A SYSTEM OF EXPONENTS WITH DEGENERATING COEFFICIENTS

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ABSTRACT. A system of exponents of power character degenerating coefficients is considered in the paper. Under definited conditions on the coefficients the basicity criterion of this system is obtained in Lebesgue spaces.

Keywords: a system of exponents, basicity, Hardy weight class.

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1. Introduction

The paper studies basicity in Lebesgue space $L_p \equiv L_p(-\pi,\pi)$, $1<p<+\infty$, of the following system of exponents

$$\left\{A^{+}(t)\,\omega^{+}(t)\,e^{int};A^{-}(t)\,\omega^{-}(t)\,e^{-ikt}\right\}_{n\geq 0;k\geq 1},\tag{1}$$

with degenerating coefficients $\omega^{\pm}(t)$ of the form

$$\omega^{\pm}(t)\equiv |t-\pi|^{\beta_{\pi}^{\pm}}\cdot|t+\pi|^{\beta_{-\pi}^{\pm}}\cdot\prod_{k=1}^{r^{\pm}}\left|t-t_k^{\pm}\right|^{\beta_k^{\pm}},\tag{2}$$

where $\left\{t_k^{\pm}\right\}_1^{r^{\pm}}\subset R$ and $A^{\pm}(t)\equiv\left|A^{\pm}(t)\right|e^{i\alpha^{\pm}(t)}$ are complex valued functions on $[-\pi,\pi]$.

Earlier, basicity of the system $\left\{e^{i(n+\alpha signn)t}\right\}_{n\in Z}$ (Z is a set of integers) in L_p, $1<p<+\infty$, that is a particular case of (1), was completely studied in the papers [3-5]. The most general case of system (1) without degenerations of $\omega^{\pm}(t)$ was considered in [1]. Basicity of system (2) in L_p with degenerations of the form (2) in the case when $\beta_{\pi}^{\pm}=\beta_{-\pi}^{\pm}=0$, was first considered in [7].

In the present paper, we study basicity of system (1) in L_p with degenerations of the form (2).

General investigation scheme is as follows: Hardy weight classes are determined interior and exterior to a unit circle, respectively. A special boundary value problem of the theory of analytic functions in these classes is considered. Fredholm property of this problem is proved under certain conditions. Using these results, basicity of system (1) in L_p is established.

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2. Basic assumptions and notions

Accept the following denotation

$$\|f\|_{\infty} = \sup_{(-\pi,\pi)} vrai \, |f(t)|;$$

$$\|f\|_{p,\nu} = \left(\int_{-\pi}^{\pi} |f(t)|^p \nu(t) \, dt \right)^{1/p}, 1 \le p < +\infty.$$

Assume that the following main assumptions are fulfilled:

1) $A^{\pm}(t) \in L_{\infty}; \|A^{\pm}\|_{\infty}^{-1} < +\infty, \left\{ \beta_{\pi}^{\pm}; \beta_{-\pi}^{\pm}; \beta_k^{\pm}, k = \overline{1, r^{\pm}} \right\} \subset \left(-\frac{1}{p}, +\infty \right);$

2) $L^{\pm}(t)$ are piecewise Holder functions on the segment $[-\pi, \pi]$; $\{s_k\}_1^r : -\pi < s_1 < \ldots < s_r < \pi$ are discontinuity points of the function $\theta(t) \equiv \alpha^-(t) - \alpha^+(t)$.

We'll use Hardy weight classes $H_{p,\nu}^{\pm}$ introduced in the paper [6]. Suppose

$$\tilde{H}^+ \equiv \left\{ f \in H_1^+ : f^+ \in L_{p,\nu} \right\},$$

where $H_1^{\pm}$ are ordinary Hardy classes interior and exterior to a unit circle, respectively, ν is some weight function, $L_{p,\nu}$ is Lebesgue's ordinary weight class on $[-\pi, \pi]$, $f^{\pm}\left(e^{it}\right)$ are non-tangential boundary values of the function $f \in H_1^{\pm}$ on a unit circle. Introduce on $\tilde{H}^+$ the norm $\| \cdot \|_{\tilde{H}^+}$ as follows

$$\|f\|_{\tilde{H}^+} \equiv \left\| f^+\left(e^{it}\right) \right\|_{p,\nu}, f \in \tilde{H}^+. \tag{3}$$

The following statement is valid

Statement 1. *Let $|\nu|^{-\frac{p}{q}} \in L_1$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 \le p < +\infty$. Then the space $\tilde{H}^+$ with respect to the norm* (3) *becomes Banach and we denote it by $H_{p,\nu}^+$.*

The class ${}_mH_{p,\nu}^-$ is introduced in the same way. Let ${}_mH_1^-$ be a Hardy class of analytic exterior to a unit circle functions possessing at most m order at infinity. Accept

$$\tilde{H}^- \equiv \left\{ f \in {}_mH_1^- : f^-\left(e^{it}\right) \in L_{p,\nu} \right\}.$$

The similar statement holds in this case as well.

Statement 2. *Let $|\nu|^{-\frac{p}{q}} \in L_1$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 \le p < +\infty$. Then the space $\tilde{H}^-$ with respect to the norm*

$$\|f\|_{\tilde{H}^-} = \left\| f^-\left(e^{it}\right) \right\|_{p,\nu}, f \in \tilde{H}^-,$$

is Banach and we denote it by ${}_mH_{p,\nu}^-$.

The following conjugation problem in the class $H_{p,\nu^+} \times {}_mH_{p,\nu^-}$

$$F^+(\tau) + G(\tau) F^-(\tau) = g(\tau), \tau \in \Gamma, \tag{4}$$

where $\Gamma : |\tau| = 1$ is a unit circle, plays an important part in proving basicity. As a solution of problem (4) we take the pair $(F^+(z); F^-(z)) \in H_{p,\nu}^+ \times {}_mH_{p,\nu^-}^-$ whose non-tangential boundary values almost everywhere on Γ satisfy relation (4), where $g \in L_p(\Gamma)$ is the right hand side, $G(\tau)$ is the problem's coefficient. We should consider the specific case

$$G\left(e^{it}\right) = \frac{A^-(t)\,\omega^-(t)}{A^+(t)\,\omega^+(t)}; \; \nu^{\pm}(t) \equiv \left|\omega^{\pm}\right|^p.$$

The following analytic introduced interior (sign "+") and exterior (sign "-") to a unit circle functions $X_k^{\pm}(z)$, $k = \overline{1, 3}$ are

$$X_1^{\pm}(z) = \exp\left\{ \pm \frac{1}{4\pi} \int_{-\pi}^{\pi} \ln \frac{\omega^-(t)}{\omega^+(t)} \cdot \frac{e^{it} + z}{e^{it} - z} dt \right\},$$

$$X_2^{\pm}(z) = \exp\left\{\pm\frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\left|\frac{A^-(t)}{A^+(t)}\right|\cdot\frac{e^{it}+z}{e^{it}-z}dt\right\},$$

$$X_3^{\pm}(z) = \exp\left\{\pm\frac{i}{4\pi}\int_{-\pi}^{\pi}\theta(t)\cdot\frac{e^{it}+z}{e^{it}-z}dt\right\}.$$

Determine

$$Z_k(z) \equiv \begin{cases} X_k^+(z), & |z|<1, \\ \left[X_k^-(z)\right]^{-1}, & |z|>1, \ \ k=\overline{1,3}. \end{cases}$$

Assume

$$Z(z) \equiv \prod_{k=1}^{3} Z_k(z).$$

3. General solution

Consider the following homogeneous problem

$$F^+(\tau) + G(\tau) F^-(\tau) = 0, \tau \in \Gamma. \qquad (5)$$

Under definited conditions on the coefficient $G(\tau)$ we can obtain representation for more general solution of homogeneous problem (5). Thus, by Sokhotskiy-Plemel formulas we have

$$G(\tau) = \frac{Z^+(\tau)}{Z^-(\tau)}, \tau \in \Gamma.$$

Considering this expression in (5), we get

$$\frac{F^+(\tau)}{Z^+(\tau)} = -\frac{F^-(\tau)}{Z^-(\tau)}, \tau \in \Gamma.$$

Introduce the following piecewise-analytic function $\Phi(z)$

$$\Phi(z) \equiv \begin{cases} \frac{F^+(z)}{Z^+(z)}, & |z|<1, \\ -\frac{F^-(z)}{Z^-(z)}, & |z|>1. \end{cases}$$

Consequently

$$\Phi^+(\tau) = \Phi^-(\tau), \tau \in \Gamma.$$

If $\Phi \in H_1^+ \times {}_mH_1^-$, it follows from the uniqueness theorem [7] that $\Phi(z)$ is a polynomial of at most m power. It suffices to prove $\Phi^{\pm}(\tau) \in L_1(\Gamma)$. Since the expression $F^-\left(e^{it}\right)\cdot\omega^-(t)$ belongs to the space L_p (it follows from definition of the solution), it suffices to show that the expression $\left[\omega^-(t)\cdot Z^-\left(e^{it}\right)\right]^{-1}$ belongs to L_q. Then, the required one will follow from Holder's inequality.

So, in $\left\{h_k^+\right\}$ we refer all positive numbers among h_k, $k=\overline{1,r}$. Absolute values of the remaining h_k denote by h_k^-. Their appropriate jump points denote by $\left\{s_k^+\right\}$ and $\left\{s_k^-\right\}$. Introduce

$$u_0(t) = \left\{\sin\left|\frac{t-\pi}{2}\right|\right\}^{-\frac{h^{(0)}}{\pi}}\cdot\exp\left\{-\frac{1}{4\pi}\right\}\int_{-\pi}^{\pi}\theta_0(s)\, ctg\frac{t-s}{2}ds,$$

where $\theta_0(t)$ is a continuous part of Jordan expansion. $\theta(t) = \theta_0(t) + \theta_1(t)$ $\theta_1(t)$ is jumps function and $h^{(0)}= \theta_0(\pi) - \theta_0(-\pi)$. Denote

$$u^{\pm}(t)=\prod_{k}\left\{\sin\left|\frac{t-s_k^{\pm}}{2}\right|\right\}^{\frac{h_k^{\pm}}{2\pi}}.$$

Let $h_\pi=\theta(-\pi)-\theta(\pi)$. Following the results of the paper [5], we have:

$$\left|Z_3^-\left(e^{it}\right)\right|=u_0(t)\left[u^+(t)\right]^{-1}\cdot u^-(t)\cdot\left\{\sin\left|\frac{t-\pi}{2}\right|\right\}^{-\frac{h_\pi}{2\pi}}.$$

Considering these expressions, we can easily get the following representation:

$$Y(t)\equiv\left|\omega^-(t)\cdot Z^-\left(e^{it}\right)\right|^{-1}=\left|Z_2^-\left(e^{it}\right)\right|^{-1}\cdot\left|Z_3^-\left(e^{it}\right)\right|^{-1}\cdot\left|\omega^-(t)\,\omega^+(t)\right|^{\frac{1}{2}}.$$

As it follows from the results of lemma 19.2 of the paper [7, p.195], the relations: $\|u_0\|_\infty^{\pm1}<+\infty$; $\left\|Z_2^-\left(e^{it}\right)\right\|_\infty^{\pm1}<+\infty$ are fulfilled.

For the relation $0<\delta\le\frac{f(t)}{g(t)}\le\delta^{-1}<+\infty$, as $t\to a$, we accept the standard denotation $f(t)\sim g(t)$, as $t\to a$. It is obvious that
$Y(t)\sim\left|\omega^-(t)\cdot\omega^+(t)\right|^{\frac{1}{2}}\cdot\left\{\sin\left|\frac{t-\pi}{2}\right|\right\}^{\frac{h_\pi}{2\pi}}$, as $t\to\pm\pi$.
Considering the expression $\omega^{\pm}(t)$, we get
$Y(t)\sim|t-\pi|^{\frac{\beta_\pi^++\beta_\pi^-}{2}+\frac{h_\pi}{2\pi}}$, as $t\to\pi$;
$Y(t)\sim|t+\pi|^{\frac{\beta_{-\pi}^++\beta_{-\pi}^-}{2}+\frac{h_\pi}{2\pi}}$, as $t\to-\pi$.
In order to get similar representation in the neighbourhood the other degeneration points, we must some quantities. So, let $T^{\pm}\equiv\left\{t_k^{\pm}\right\}_{k=1}^{r^{\pm}}$, $S\equiv\{s_k\}_1^r$. Accept $T^+\bigcup T^-\bigcup S=\{\sigma_k\}_1^{\ell}$. By $T_k^{\pm}$ we denote the one-element set $\left\{t_k^{\pm}\right\}$, $k=\overline{1,r^{\pm}}$. Similar denotation are accepted for $S_k=\{s_k\}$ and $\Omega_k=\{\sigma_k\}$. Consider the following set function

$$\chi(A)=\begin{cases}1, & A\neq 0,\\ 0, & A=\emptyset,\end{cases}$$

where $\emptyset$ is an empty set. Accept

$$\left.\begin{array}{ll}\lambda_k^{\pm}=\frac{1}{2}\sum_{i=1}^{r^{\pm}}\beta_i^{\pm}\chi\left(T_i^{\pm}\bigcap\Omega_k\right), & k=\overline{1,\ell}\\ \lambda_k=\frac{1}{2\pi}\sum_{i=1}^{r}h_i\chi\left(S_i\bigcap\Omega_k\right), & k=\overline{1,\ell}\end{array}\right\}\tag{6}$$

Assume

$$\gamma_k=\lambda_k^++\lambda_k^-+\lambda_k,\qquad k=\overline{1,\ell}.\tag{7}$$

In these denotation, the expression $\left|\omega^-(t)\cdot\omega^+(t)\right|^{\frac{1}{2}}$ in the neighbourhood the points σ_k is of the form: $\left|\omega^-(t)\cdot\omega^+(t)\right|^{\frac{1}{2}}\sim|t-\sigma_k|^{\lambda_k^++\lambda_k^-}$, as $t\to\sigma_k$, $k=\overline{1,\ell}$. On the other hand,
$\left|Z_3^-\left(e^{it}\right)\right|^{-1}\sim\left\{\sin\left|\frac{t-\sigma_k}{2}\right|\right\}^{\lambda_k}\sim|t-\sigma_k|^{\lambda_k}$, as $t\to\sigma_k$, $k=\overline{1,\ell}$
is valid.

As a result, for $Y(t)$ we obtain
$Y(t)\sim\ |t-\sigma_k|^{\gamma_k}$, as $t\to\sigma_k$, $k=\overline{1,\ell}$.

It follows from these relations that if the inequalities

$$-\frac{1}{q}<\gamma_k,k=\overline{1,\ell};$$

$$-\frac{2}{q}<\beta_\pi^++\beta_\pi^-+\frac{h_k}{\pi};-\frac{2}{q}<\beta_{-\pi}^++\beta_{-\pi}^-+\frac{h_k}{\pi},$$

are fulfilled, then $Y(t)$ belongs to L_q. Then it follows from arguments mentioned above that $\Phi(z)$ is a polynomial $P_m(z)$ of at most m order, i.e. $F(z) = Z(z) \cdot P_m(z)$. It is clear that if $F(z) \in H^+_{p,\nu^+} \times {}_mH^-_{p,\nu^-}$, then $F(z)$ is a solution of homogeneous problem (5). It suffices to show that $Y^-(t) = |Z^-|\left(e^{it}\right) \cdot \omega^-(t)$ belongs to L_p. It is easy to see that if

$$\gamma_k < \frac{1}{p}, k = \overline{1,\ell};$$

$$\beta^+_\pi + \beta^-_\pi + \frac{h_k}{\pi} < \frac{2}{p}; \beta^+_{-\pi} + \beta^-_{-\pi} + \frac{h_k}{\pi} < \frac{2}{p}$$

hold, then $Y^{-1}(t)$ belongs to L_p. Thus, it is valid the following

Theorem 1. *Let conditions 1); 2) hold and the inequalities*

$$-\frac{1}{q} < \gamma_k < \frac{1}{p}, k = \overline{1,\ell};$$

$$-\frac{2}{q} < \beta^+_\pi + \beta^-_\pi + \frac{h_k}{\pi} < \frac{2}{p}; -\frac{2}{q} < \beta^+_{-\pi} + \beta^-_{-\pi} + \frac{h_k}{\pi} < \frac{2}{p},$$

where the quantities $\{\gamma_k\}_1^\ell$ *are determined by expressions* (6), (7), *be fulfilled. Then a general solution of homogeneous problem* (5) *in the class* $H^+_{p,\nu^+} \times {}_mH^-_{p,\nu^-}$ *is of the form*

$$F(z) = Z(z) \cdot P_m(z),$$

where $Z(z)$ *is a canonic solution,* $P_m(z)$ *is an arbitrary polynomial of at most* m *power.* This theorem yields the

Corollary 1. *Let all the conditions of theorem 1 be fulfilled. Then homogeneous problem* (5) *under condition* $F(\infty) = 0$ *in the class* $H^+_{p,\nu^+} \times {}_mH^-_{p,\nu^-}$ *only is trivially solvable.*

4. Basicity

Studying the basicity of system (1) in L_p we'll follow the paper [6]. Take $\forall f \in L_p$ and consider the homogeneous problem

$$\begin{cases} F^+(\tau) + G(\tau) F^-(\tau) = g(\tau), & t \in \Gamma, \\ F(\infty) = 0 \end{cases} \tag{8}$$

in the class $H^+_{p,\nu^+} \times {}_mH^-_{p,\nu^-}$, where $g\left(e^{it}\right) \equiv \frac{f(t)}{\omega^+(t)}$, $\forall t \in [-\pi, \pi]$. Assume that all the conditions of the Theorem 1 are fulfilled. Then it is clear that the homogeneous problem has only a trivial solution and this means that homogeneous problem (8) may have a unique solution. Show that it is solvable. Let $Z(z)$ be any canonic solution of the homogeneous problem. Consider the following piecewise-analytic function $F(z)$:

$$F(z) = \frac{Z(z)}{2\pi} \int_{-\pi}^{\pi} \frac{g\left(e^{i\sigma}\right)}{Z^+\left(e^{i\sigma}\right)} \cdot \frac{d\sigma}{1 - e^{i(t-\sigma)}}.$$

If follows from the Sokhotskiy-Plemel formula that

$$F^+\left(e^{it}\right)\omega^+\left(e^{it}\right) = f(t) + \frac{Z_0(t)}{2\pi} \int_{-\pi}^{\pi} \frac{f(\sigma)}{Z_0(\sigma)} \frac{d\sigma}{1 - e^{i(t-\sigma)}}, \tag{9}$$

where $Z_0(t) \equiv \omega^+(t) \cdot Z^+\left(e^{it}\right)$. As it follows from the theorem 16.5, monography [2], the integral operator

$$I[f] = Z_0(t) \int_{-\pi}^{\pi} \frac{f(\sigma)}{Z_0(\sigma)} \cdot \frac{d\sigma}{1 - e^{i(t-\sigma)}}$$

boundedly acts from L_p to L_p. Then it follows from expression (9) that $F^+\left(e^{it}\right)$ belongs to L_{p,ν^+}. As we have shown $Y^{-1}(t)$ belongs to L_q. Thus it follows from the obvious relation

$$A^+(t)\,\omega^+(t)\,Z^+\left(e^{it}\right)=-A^-(t)\,\omega^-(t)\,Z^-\left(e^{it}\right),$$

that $Z_0(t)\in L_q$. Consequently, $\frac{f(t)}{Z_0(\sigma)}\in L_1$ and this means that the Cauchy-Lebesgue type integral $\int_{-\pi}^{\pi}\frac{f(\sigma)}{Z_0(\sigma)}\frac{d\sigma}{1-z}$ belongs to the class H_1^+. From expression then we have $F(z)$ that it belongs to H_μ^+ for some $\mu>0$. Further we'll assume that the inequalities:

$$-\frac{1}{p}<\beta_\pi^\pm<\frac{1}{q};-\frac{1}{p}<\beta_{-\pi}^\pm<\frac{1}{q};-\frac{1}{p}<\beta_k^\pm<\frac{1}{q},k=\overline{1,r^\pm}. \tag{10}$$

hold. Obviously $|\omega^\pm(t)|^{-1}\in L_q$. As the expression $F^+\left(e^{it}\right)\omega^+(t)$ belongs to L_p then it follows directly from $F^+\left(e^{it}\right)=\left[F^+\left(e^{it}\right)\omega^+(t)\right]\cdot|\omega^+(t)|^{-1}$ that $F^+(z)\in H_1^+$.

Finally, we get that $F^+(z)\in H^+_{p,\nu^+}$. From the same reasoning's it follows that $F^-(z)\in {}_mH^-_{p,\nu^-}$. Since $z(\infty)=1$, it is clear that $F(\infty)=1$. Thus, $F(z)$ is a solution of problem (8). Let L_p^+ and ${}_mL_p^-$ be contractions of the functions from the classes H_p^+ and ${}_mH_p^-$ on a unit circle, respectively. Denote

$$L^+_{p,\nu}\equiv\left\{f\in L_1^+:\|f\|_{p,\nu}<+\infty\right\},$$

$${}_mL^-_{p,\nu}\equiv\left\{f\in L_1^-:\|f\|_{p,\nu}<+\infty\right\}.$$

As it is shown in the paper [6], if inequalities (10) are fulfilled, the system $\left\{e^{int}\right\}_{n\geq0}(\left\{e^{-int}\right\}_{n\geq m})$ makes a basis on $L^+_{p,\nu^+}({}_mL^-_{p,\nu^-})$. It is obvious that $F^+\left(e^{it}\right)\in L^+_{p,\nu^+}$ $(F^-\left(e^{it}\right)\in{}_{-1}L^-_{p,\nu^-})$. Expanding these functions in appropriate bases, we get

$$f(t)=A^+(t)\,\omega^+(t)\sum_{n=0}^{\infty}f_n^+e^{int}+A^-(t)\,\omega^-(t)\sum_{n=1}^{\infty}f_n^-e^{-int}, \tag{11}$$

where $F^+\left(e^{it}\right)=\sum_{n=0}^{\infty}f_n^+e^{int}$ $(F^{+-}\left(e^{it}\right)=\sum_{n=1}^{\infty}f_n^-e^{-int})$.

Convergence is understood in L^+_{p,ν^+} $({}_{-1}L^-_{p,\nu^-})$.

We directly get from (11) that an arbitrary function from L_p is expanded in it in system (1). For basicity of system (1) in L_p, it suffices to show its minimality in it.

In problem (8), in the place of $f(t)$ we take the function $f(t)\equiv A^+(t)\,\omega^+(t)\,e^{in_0t}$, where $n_0\geq0$ is some fixed integer. We have

$$\left\{\begin{array}{l} A^+(t)\,\omega^+(t)\,F^+\left(e^{it}\right)+A^-(t)\,\omega^-(t)\,F^-\left(e^{it}\right)=A^+(t)\,\omega^+(t)\,e^{in_0t},\\ t\in[-\pi,\pi]\,;F(\infty)=0\end{array}\right. \tag{12}$$

It is obvious that the function

$$F_0(z)=\left\{\begin{array}{ll} z^{n_0}, & |z|>1,\\ 0, & |z|<1,\end{array}\right.$$

is a solution of the problem in $H^+_{p,\nu^+}\times{}_mH^-_{p,\nu^-}$. On the other hand, as we have shown, the function

$$F(z)=\frac{Z(z)}{2\pi}\int_{-\pi}^{\pi}\frac{A^+(t)\,e^{in_0t}}{Z^+\left(e^{it}\right)}\cdot\frac{dt}{1-ze^{-it}}$$

is also a solution of (12). Consequently,

$$\frac{Z(z)}{2\pi}\int_{-\pi}^{\pi}\frac{A^{+}(t)\,e^{in_0t}}{Z^{+}(e^{it})}\cdot\frac{dt}{1-ze^{-it}}=\begin{cases} z^{n_0}, & |z|>1\\ 0, & |z|<1\end{cases}$$

is valid. Having expanded the left hand side in powers of z^n and denoted the appropriate integrand expression at the coefficient z^n by $h_n^{+}(t)$, from the previous relation we obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}A^{+}(t)\,\omega^{+}(t)\,e^{in_0t}\left[\frac{1}{\omega^{+}(t)\cdot Z^{+}(e^{it})}\frac{Z(z)}{1-ze^{-it}}\right]dt=$$
$$=\sum_{n=0}^{\infty}\int_{-\pi}^{\pi}A^{+}(t)\,\omega^{+}(t)\,e^{in_0t}\overline{h_n^{+}(t)}dtz^n=\begin{cases} z^{n_0}, & |z|>1,\\ 0, & |z|<1,\end{cases}$$

where $\overline{(\cdot)}$ is a complex conjugation. Hence it follows that

$$\int_{-\pi}^{\pi}A^{+}(t)\,\omega^{+}(t)\,e^{in_0t}\overline{h_n^{+}(t)}dt=\delta_{n_0n},\forall n_0,n\geq 0,$$

where δ_{nk} - is a Kronecker symbol. It follows from $\left[\omega^{+}(t)\,Z^{+}\left(e^{it}\right)\right]^{-1}\in L_q$, that $\{h_n^{+}\}_{n\geq 0}\subset L_q$.

From the same considerations we get

$$\int_{-\pi}^{\pi}A^{-}(t)\,\omega^{-}(t)\,e^{-ikt}\overline{h_n^{+}(t)}dt=0,\forall k\geq 1;\forall n\geq 0.$$

In the same way, it is proved that there is such a system $\{h_n^{-}\}_{n\geq 1}\subset L_q$ for which the relations are valid

$$\int_{-\pi}^{\pi}A^{-}(t)\,\omega^{-}(t)\,e^{-ikt}\overline{h_n^{-}(t)}dt=\delta_{nk},\forall n,k\geq 1;$$
$$\int_{-\pi}^{\pi}A^{+}(t)\,\omega^{+}(t)\,e^{ikt}\overline{h_n^{-}(t)}dt=0,\forall k\geq 0,\forall k\geq 1.$$

This proves minimality of the system (1) in L_p. Thus, we proved the following theorem.

Theorem 2. *Let conditions 1); 2) hold, inequalities* (10) *be valid and*

$$-\frac{1}{q}<\gamma_k<\frac{1}{p},k=\overline{1,\aleph};$$
$$-\frac{2}{q}<\beta_{\pi}^{+}+\beta_{\pi}^{-}+\frac{h_{\pi}}{\pi}<\frac{2}{p};-\frac{2}{q}<\beta_{-\pi}^{+}+\beta_{-\pi}^{-}+\frac{h_{\pi}}{\pi}<\frac{2}{p},$$

where the quantities $\{\gamma_k\}_1^{\aleph}$ *are determined by expressions* (6), (7), *be fulfilled. Then the system* (1) *forms a basis in* L_p, $1<p<+\infty$.

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